

# Problem 2.63

[Difficulty: 4]

**2.63** The thin outer cylinder (mass  $m_2$  and radius  $R$ ) of a small portable concentric cylinder viscometer is driven by a falling mass,  $m_1$ , attached to a cord. The inner cylinder is stationary. The clearance between the cylinders is  $a$ . Neglect bearing friction, air resistance, and the mass of liquid in the viscometer. Obtain an algebraic expression for the torque due to viscous shear that acts on the cylinder at angular speed  $\omega$ . Derive and solve a differential equation for the angular speed of the outer cylinder as a function of time. Obtain an expression for the maximum angular speed of the cylinder.

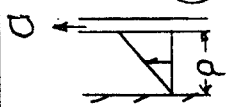
Solution:

Basic equations:  $\tau = \mu \frac{du}{dy}$

$$\Sigma F = ma, \quad \Sigma M = I\alpha$$

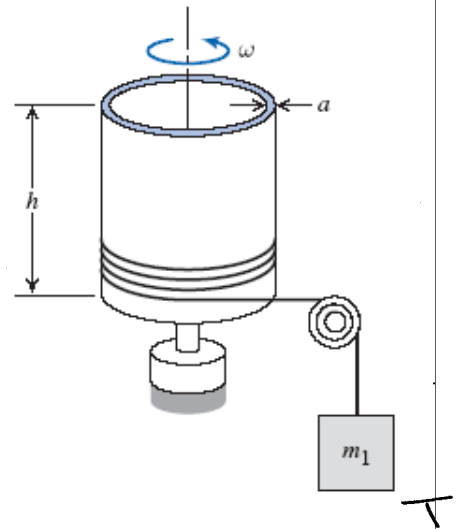
Assume: (1) Newtonian fluid  
(2) linear velocity profile

In the gap,  $\tau = \mu \frac{du}{dy} = \mu \frac{U}{a} = \frac{\mu R \omega}{a}$

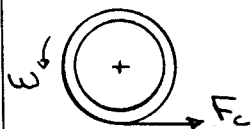


$$T = \tau R = \frac{\mu R \omega}{a} (2\pi R h) R$$

$$T = \frac{2\pi R^3 \mu h}{a} \omega$$



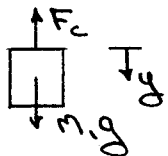
During acceleration, let the tension in the cord be  $F_c$



For the cylinder  $\Sigma M = F_c R - T = I\alpha = m_2 R^2 \frac{d\omega}{dt} \dots (1)$

For the mass  $\Sigma F_y = m_1 g - F_c = m_1 a = m_1 \frac{dv}{dt} = m_1 R \frac{d\omega}{dt} \dots (2)$

$$\therefore F_c = m_1 g - m_1 R \frac{d\omega}{dt}$$



Substituting into eq. (1)

$$m_1 g R - \frac{2\pi R^3 \mu h}{a} \omega = (m_1 + m_2) R^2 \frac{d\omega}{dt}$$

Let  $m_1 g R = b$ ,  $-2\pi R^3 \mu h / a = c$ ,  $(m_1 + m_2) R^2 = f$

Then,  $b + c\omega = f \frac{d\omega}{dt}$  or  $\int_0^t \frac{1}{f} dt = \int_0^\omega \frac{d\omega}{(b+c\omega)}$

Integrating,  $\frac{1}{f} t = \frac{1}{c} \ln(b+c\omega) \Big|_0^\omega = \frac{1}{c} \ln \frac{(b+c\omega)}{b} = \frac{1}{c} \ln \left(1 + \frac{c}{b} \omega\right)$

$$\frac{c}{f} t = \ln \left(1 + \frac{c}{b} \omega\right) \Rightarrow e^{\frac{c}{f} t} = \left(1 + \frac{c}{b} \omega\right) \Rightarrow \omega = \frac{b}{c} (e^{\frac{c}{f} t} - 1)$$

Substituting for  $b, c$ , and  $f$

$$\omega = \frac{m_1 g R a}{2\pi R^3 \mu h} \left(1 - e^{-\frac{2\pi R^3 \mu h}{a(m_1 + m_2) R^2} t}\right) = \frac{m g a}{2\pi R^2 \mu h} \left[1 - e^{-\frac{2\pi R^3 \mu h}{a(m_1 + m_2) R^2} t}\right]$$

Maximum  $\omega$  occurs at  $t \rightarrow \infty$

$$\omega_{\max} = \frac{m g a}{2\pi R^2 \mu h}$$

$\omega_{\max}$